

THE f -STABILITY INDEX OF THE CONSTANT WEIGHTED MEAN CURVATURE HYPERSURFACES IN GRADIENT RICCI SOLITONS

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ABSTRACT. In this paper, we prove that a noncompact complete hypersurface with finite weighted volume, weighted mean curvature vector bounded in norm, and isometrically immersed in a complete weighted manifold is proper. In addition, we obtain an estimate for f -stability index of a constant weighted mean curvature hypersurface with finite weighted volume and isometrically immersed in a shrinking gradient Ricci soliton that admits at least one parallel field globally defined. For such hypersurface, we still give a necessary condition for equality to be achieved in the estimate obtained.

1. INTRODUCTION

Consider a $(n + 1)$ -dimensional Riemannian manifold (\overline{M}^{n+1}, g) endowed with a weighted measure of the form $e^{-f}d\mu$, where f is a smooth function on M and $d\mu$ is the volume element induced by the Riemannian metric g . A *weighted manifold* is a triple

$$\overline{M}_f^{n+1} = (\overline{M}^{n+1}, g, e^{-f}d\mu).$$

A natural extension of the Ricci tensor to this new context is the Bakry-Émery Ricci tensor, see [1], given by

$$\overline{Ric}_f = \overline{Ric} + \overline{\nabla}^2 f,$$

where $\overline{\nabla}^2 f$ is the Hessian of f on \overline{M}^{n+1} . It is known that a complete weighted manifold satisfying $\overline{Ric}_f \geq kg$ for some constant $k > 0$ is not necessarily compact. One of the examples is the Gaussian shrinking soliton $\left(\mathbb{R}^{n+1}, g_{can}, e^{-\frac{|x|^2}{4}}d\mu\right)$ with the canonical metric g_{can} and $\overline{Ric}_f = \frac{1}{2}g_{can}$.

The gradient Ricci solitons are natural generalizations of the Einstein metrics and was introduced by Hamilton in [17]. Indeed, a complete Riemannian metric g on a smooth manifold \overline{M}^{n+1}

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is a *shrinking gradient Ricci soliton* if there exists a potential function f , and a real constant $k > 0$ such that the Ricci tensor \overline{Ric} of the metric g satisfies the equation

$$(1.1) \quad \overline{Ric} + \overline{\nabla}^2 f = kg,$$

where $\overline{\nabla}^2 f$ denotes the Hessian of f . In this context, the gradient Ricci solitons are complete weighted manifolds with $\overline{Ric}_f = kg$ for some constant real k .

Observe that when the potential function is a constant, the gradient Ricci solitons are simply Einstein metrics. It is still important to mention that gradient Ricci solitons plays an important role in Hamilton's Ricci flow and correspond to self-similar solutions, and often arise as Type I singularity models. For more details see [11].

Let $x : M^n \rightarrow \overline{M}_f^{n+1}$ be an isometric immersion of a Riemannian orientable manifold M^n into weighted manifold \overline{M}_f^{n+1} . The function $f : \overline{M} \rightarrow \mathbb{R}$, restricted to M , induces a weighted measure $e^{-f}d\sigma$ on M . Thus, we have an induced weighted manifold $M_f^n = (M, \langle, \rangle, e^{-f}d\sigma)$.

The *second fundamental form* A of x is defined by

$$A(X, Y) = (\overline{\nabla}_X Y)^\perp, \quad X, Y \in T_p M, \quad p \in M,$$

where \perp symbolizes the projection above the normal bundle of M . The *weighted mean curvature vector* of M is defined by

$$\mathbf{H}_f = \mathbf{H} + (\overline{\nabla} f)^\perp,$$

and its the *weighted mean curvature* H_f is given by

$$\mathbf{H}_f = -H_f \eta,$$

where $\mathbf{H} = \text{tr} A$ and η is unit outside normal vector field. The hypersurface M is called *f -minimal* when its weighted mean curvature vector \mathbf{H}_f vanishes identically, and when there exists real constant C such that $\mathbf{H}_f = -C\eta$, we say the hypersurface M has *constant weighted mean curvature*.

The *weighted volume* of a measurable set $\Omega \subset M$ is given by

$$\text{Vol}_f(\Omega) = \int_{\Omega} e^{-f} d\sigma.$$

Let $B_r^{\overline{M}}$ be the geodesic ball of \overline{M} with center in a fixed point $o \in \overline{M}$ and radius $r > 0$. It is said that the weighted volume of M has *polynomial growth* if there exists positive numbers α and C such that

$$(1.2) \quad \text{Vol}_f(B_r^{\overline{M}} \cap M) \leq Cr^\alpha$$

for any $r \geq 1$. When $\alpha = n$ in (1.2), M is said to have Euclidean volume growth.

We can consider either f -minimal or constant weighted mean curvature hypersurfaces in gradient Ricci solitons. In particular, a self-shrinker to the mean curvature flow is a f -minimal hypersurface of the shrinking Gaussian soliton. In [4], Cheng and Zhou showed that for f -minimal hypersurfaces in the shrinking Gaussian soliton $(\mathbb{R}^{n+1}, g_{can}, e^{-\frac{|x|^2}{4}} dx)$, the properness of its immersion, its polynomial volume growth, and its finite weighted volume are equivalent to each other. Those equivalences are still being valid for f -minimal hypersurfaces immersed in a complete shrinking gradient Ricci soliton \overline{M}_f satisfying $\overline{Ric}_f = \frac{1}{2}g$, where g is Riemannian metric and f is a convex function (see [2]). In this direction, the following result was obtained:

Proposition 1.1. *Let M^n be a noncompact complete hypersurface isometrically immersed in a complete weighted manifold \overline{M}_f^m , with weighted mean curvature vector bounded in norm. If M^n has finite weighted volume, then M^n is proper.*

Proposition 1.2. *Let $f \in C^\infty(\overline{M})$ be a convex function, \overline{M}_f^m a gradient Ricci soliton with $\overline{Ric}_f = \frac{1}{2}g$, and $x : M^n \rightarrow \overline{M}_f^m$ a noncompact complete immersion with weighted mean curvature vector satisfying*

$$\sup_{x \in M} \langle \mathbf{H}_f, \overline{\nabla} f \rangle < \infty.$$

If M^n is proper, then it has finite weighted volume and Euclidean volume growth.

Now it is known that the *weighted Laplacian operator* Δ_f , defined by

$$\Delta_f u := \Delta u - \langle \nabla f, \nabla u \rangle,$$

is associated to $e^{-f} d\sigma$ as well as Δ is associated to $d\sigma$. Moreover, Δ_f is a self-adjoint operator on the L_f^2 space of square integrable functions on M with respect to the measure $e^{-f} d\sigma$, and therefore, the L_f^2 spectrum of Δ_f on M , denoted by $\sigma(-\Delta_f)$, is a subset of $[0, +\infty)$.

Next, let $F : (-\varepsilon, \varepsilon) \times M \rightarrow \overline{M}_f$, $F_f(p) = F(t, p)$ for all $t \in (-\varepsilon, \varepsilon)$ and $p \in M$ be a variation of the immersion x associated with the normal vector field $u\eta$, where $u \in C_c^\infty(M)$. The corresponding variation of the *functional weighted area* $\mathcal{A}_f(t) = \text{Vol}_f(F_t(M))$ satisfies

$$(1.3) \quad \mathcal{A}'_f(0) = \int_M H_f u e^{-f} d\sigma,$$

where H_f is such that $\mathbf{H}_f = -H_f \eta$. The expression (1.3) is known as *first variation formula*.

The f -minimal hypersurfaces are critical points of the functional weighted area. Yet, the hypersurfaces with constant weighted mean curvature can be viewed as critical points of the functional weighted area restricted to variations which preserve the *enclose weighted volume*,

i.e., for functions $u \in C_c^\infty(M)$ which satisfy the additional condition

$$\int_M u e^{-f} d\sigma = 0.$$

For such critical points, the *second variation* of the functional weighted area is given by

$$\mathcal{A}_f''(0) = - \int_M (u \Delta_f u + (|A|^2 + \overline{Ric}_f(\eta, \eta)) u^2) d\sigma,$$

where \overline{Ric}_f is the Bakry-Émery Ricci curvature and A is the second fundamental form. For more details, see [10].

Remark 1.1. *When f is a constant function, the first and second variation formula were given by Barbosa and do Carmo [7] and Barbosa, do Carmo and Eschenburg [6].*

The operator

$$L_f = \Delta_f + |A|^2 + \overline{Ric}_f(\eta, \eta)$$

is called the *f-stability operator* of the immersion x . In the f -minimal case, the f -stability operator is viewed as acting on $\mathcal{F} = C_c^\infty(M)$; in the case of the hypersurfaces with constant weighted mean curvature, the f -stability operator is viewed as acting on

$$\mathcal{F} = C_c^\infty(M) \cap \left\{ u \in C_c^\infty(M); \int_M u e^{-f} d\sigma = 0 \right\}.$$

Associated with L_f is the quadratic form

$$I_f(u, u) = - \int_M u L_f u e^{-f} d\sigma.$$

For each compact domain $\Omega \subset M$, define the index, $Ind_f \Omega$, of L_f in Ω as the maximal dimension of a subspace of \mathcal{F} where I_f is a negative definite. The *index*, $Ind_f M$, of L_f in M (or simply, the index of M) is then defined by

$$Ind_f M = \sup_{\Omega \subset M} Ind_f \Omega,$$

where the supreme is taken over all compact domains $\Omega \subset M$. For more details, see [14] and [5].

Let $M \subset \mathbb{R}^{n+1}$ be a proper, non-planar, two-sided hypersurface satisfying $Vol_f(M) < \infty$, $H = \frac{1}{2} \langle x, \eta \rangle + C$ and $Ind_f(M) \leq n$, where H is the mean curvature, x is the position vector of \mathbb{R}^{n+1} , η is the unit normal field of the hypersurface, $Ind_f(M)$ is the f -stability index and C is a real constant. McGonagle and Ross ([8], Theorem 5.6) showed that exists a natural number i such that $n + 1 - Ind_f(M) \leq i \leq n$ e $\Sigma = \Sigma_0 \times \mathbb{R}^i$. In addition, they obtained $Ind_f(M) \geq 2$.

It is important to mention that the properness hypothesis can be removed of the Theorem 5.6 of [8]. In fact, by Proposition 1.1, the finite weighted volume implies in the properness of its immersion.

Next, we obtain an estimate for f -stability index of a constant weighted mean curvature hypersurface with finite weighted volume and isometrically immersed in a gradient Ricci soliton that admit at least one parallel field globally defined. In fact,

Theorem 1.1. *Let \overline{M}_f be a shrinking gradient Ricci soliton with $\overline{Ric}_f = kg$. Let M^n be a constant weighted mean curvature hypersurface with finite weighted volume and isometrically immersed in \overline{M}_f . Denote by $\mathcal{P}_{\overline{M}_f}$ the set of parallel fields globally defined on \overline{M}_f^{n+1} and η the unit normal field to M .*

(i) *If the unit function $1 \notin \{\langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}$,*

$$(1.4) \quad Ind_f(M) \geq \dim \mathcal{P}_{\overline{M}_f} - \dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\}.$$

(ii) *If the unit function $1 \in \{\langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}$, M is totally geodesic.*

As a consequence of Theorem 1.1, we have

Corollary 1.1. *Let M^n be a constant weighted mean curvature hypersurface with finite weighted volume and isometrically immersed in a shrinking gradient Ricci soliton \overline{M}_f . Denote by $\mathcal{P}_{\overline{M}_f}$ the set of parallel fields globally defined on \overline{M}_f^{n+1} and let η the unit normal field to M . If the unit function $1 \notin \{\langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}$ and there exist a parallel field X_0 such that $\langle X_0, \eta \rangle \not\equiv 0$, then*

$$Ind_f(M) \geq 1.$$

Moreover,

$$\dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\} = \dim \mathcal{P}_{\overline{M}_f} - 1$$

whenever $Ind_f(M) = 1$.

A necessary condition for equality to be achieved in the estimate (1.4) of Theorem 1.1, is given by

Theorem 1.2. *Let \overline{M}_f be a shrinking gradient Ricci soliton with $\overline{Ric}_f = kg$. Let M^n be a constant weighted mean curvature hypersurface isometrically immersed in \overline{M}_f . Denote by $\mathcal{P}_{\overline{M}_f}$ the set of parallel fields globally defined on \overline{M}_f^{n+1} and η the unit normal field to M . Suppose that $Vol_f(M) < \infty$, $Ind_f(M) < \infty$, $\dim \mathcal{P}_{\overline{M}_f} > 0$, and*

$$Ind_f(M) = \dim \mathcal{P}_{\overline{M}_f} - \dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\}.$$

- (i) If $\text{Ind}_f(M) = \dim \mathcal{P}_{\overline{M}_f}$, M is totally geodesic and the bottom $\mu_1(M)$ of the spectrum of f -stability operator satisfies $\mu_1(M) = -k$.
- (ii) If $\text{Ind}_f(M) \neq \dim \mathcal{P}_{\overline{M}_f}$, either M is diffeomorphic to the product of a Euclidian space with some other manifold or there is a circle action on M whose orbits are not real homologous to zero.

2. PROPERNESS AND FINITE WEIGHTED VOLUME OF A CONSTANT WEIGHTED MEAN CURVATURE HYPERSURFACE

We will begin by proving Proposition 1.1 which states that a noncompact complete hypersurface with finite weighted volume, weighted mean curvature vector bounded in norm, and isometrically immersed in a complete weighted manifold is proper. Furthermore, long after we will prove Proposition 1.2, it give some conditions to that the a proper noncompact complete hypersurface have finite weighted volume.

Proof of Proposition 1.1. We supposed that M is not proper. Thus, there exists a positive real number R such that $\overline{B}_R^{\overline{M}}(o) \cap M$ is no compact in M , where $\overline{B}_R^{\overline{M}}(o)$ denotes the closure of the $B_R^{\overline{M}}(o)$. Then, for any $a > 0$ sufficiently small with $a < 2R$, there exists a sequence $\{p_k\}$ of the points in $B_R^{\overline{M}}(o) \cap M$ with $\text{dist}_M(p_k, p_j) \geq a > 0$ for any different k and j .

Since $B_{a/2}^M(p_k) \cap B_{a/2}^M(p_j) = \emptyset$ for any $k \neq j$, we obtain $B_{a/2}^M(p_j) \subset B_{2R}^{\overline{M}}(o)$, where $B_{a/2}^M(p_k)$ and $B_{a/2}^M(p_j)$ denote the intrinsic balls of M of radius $a/2$, center in p_k and p_j , respectively.

Let $\{e_1, e_2, \dots, e_n\}$ be a orthonormal basis of $T_x M$. If $x \in B_{a/2}^M(p_j)$, then the function extrinsic distance to p_j , denoted by $r_j(x) = \text{dist}_{\overline{M}}(x, p_j)$, satisfies

$$\begin{aligned}
 \overline{\nabla}^2 r_j(e_i, e_i) &= \langle \overline{\nabla}_{e_i} \overline{\nabla} r_j, e_i \rangle = \langle \overline{\nabla}_{e_i} \nabla r_j, e_i \rangle + \langle \overline{\nabla}_{e_i} (\overline{\nabla} r_j)^\perp, e_i \rangle \\
 &= \langle \nabla_{e_i} \nabla r_j, e_i \rangle + \langle (\overline{\nabla}_{e_i} \nabla r_j)^\perp, e_i \rangle - \langle A(e_i, e_i), \overline{\nabla} r_j \rangle \\
 &= \nabla^2 r_j(e_i, e_i) - \langle \mathbf{H}, \overline{\nabla} r_j \rangle.
 \end{aligned}$$

Observe that \overline{M} has bounded locally geometry, this is, there exists positive real numbers k and i_0 so that the sectional curvature of \overline{M} is bounded above by k and the injectivity radius of \overline{M} is bounded below by i_0 in a neighborhood of a point $o \in \overline{M}$. By choosing $R > 0$ such that $2R < \min\{i_0, 1/\sqrt{k}\}$, it follows from Hessian comparison of the distance (see Lemma 7.1, [3]), that

$$\overline{\nabla}^2 r_j(e_i, e_i) \geq -\sqrt{k} + \frac{1}{r_j} |e_i - \langle e_i, \overline{\nabla} r_j \rangle \overline{\nabla} r_j|^2$$

in $\overline{B_{2R}^M}(o)$. Hence, in $\overline{B_{2R}^M}(o) \cap M$,

$$\begin{aligned}
 \Delta r_j &= \sum_{i=1}^n \nabla^2 r_j(e_i, e_i) = \sum_{i=1}^n \overline{\nabla}^2 r_j(e_i, e_i) + \langle \mathbf{H}, \overline{\nabla} r_j \rangle \\
 &\geq \sum_{i=1}^n \left(-\sqrt{k} + \frac{1}{r_j} |e_i - \langle e_i, \overline{\nabla} r_j \rangle \overline{\nabla} r_j|^2 \right) + \langle \mathbf{H}, \overline{\nabla} r_j \rangle + \langle (\overline{\nabla} f)^\perp, \overline{\nabla} r_j \rangle - \langle (\overline{\nabla} f)^\perp, \overline{\nabla} r_j \rangle \\
 &= -n\sqrt{k} + \frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} + \langle \mathbf{H}_f, \overline{\nabla} r_j \rangle - \langle (\overline{\nabla} f)^\perp, \overline{\nabla} r_j \rangle \\
 &\geq -n\sqrt{k} + \frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} - |\mathbf{H}_f| - |(\overline{\nabla} f)^\perp|.
 \end{aligned}$$

By hypothesis, the norm of \mathbf{H}_f is bounded above and $\sup_{p \in \overline{B_{2R}^M}(o)} |\overline{\nabla} f(p)| < \infty$ for each $R > 0$. Thus,

$$\begin{aligned}
 \Delta r_j &\geq -n\sqrt{k} + \frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} - \sup_{p \in \overline{B_{2R}^M}(o) \cap M} |\mathbf{H}_f(p)| - \sup_{p \in \overline{B_{2R}^M}(o) \cap M} |(\overline{\nabla} f(p))^\perp| \\
 &\geq \frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} - C,
 \end{aligned}$$

where

$$C = n\sqrt{k} + \sup_{p \in \overline{B_{2R}^M}(o) \cap M} |\mathbf{H}_f(p)| + \sup_{p \in \overline{B_{2R}^M}(o)} |\overline{\nabla} f(p)|.$$

Therefore, in $\overline{B_{2R}^M}(o) \cap M$,

$$\Delta r_j^2 = 2r_j \Delta r_j + 2|\nabla r_j|^2 \geq 2r_j \left(\frac{n}{r_j} - \frac{|\nabla r_j|^2}{r_j} - C \right) + 2|\nabla r_j|^2 = 2n - 2Cr_j.$$

By choosing $a < \min\{2n/C, 2R\}$, we have for $0 < \zeta \leq a/2$,

$$\begin{aligned}
 \int_{B_\zeta^M(p_j)} (2n - 2Cr_j) d\sigma &\leq \int_{B_\zeta^M(p_j)} \Delta r_j^2 d\sigma = \int_{\partial B_\zeta^M(p_j)} \langle \nabla r_j^2, \nu \rangle d\sigma \\
 (2.1) \qquad \qquad \qquad &\leq \int_{\partial B_\zeta^M(p_j)} 2r_j |\nabla r_j| |\nu| dA \leq \int_{\partial B_\zeta^M(p_j)} 2r_j dA \\
 &\leq 2\zeta A_j(\zeta),
 \end{aligned}$$

where ν denotes the unit normal vector field pointing out of $\partial B_\zeta^M(p_j)$ and $A_j(\zeta)$ denotes the area of $\partial B_\zeta^M(p_j)$. By using co-area formula in (2.1), we have

$$\begin{aligned} \int_{B_\zeta^M(p_j)} (n - Cr_j) d\sigma &= \int_0^\zeta \int_{\partial B_t^M(p_j)} (n - Cr_j) |\nabla r_j|^{-1} dA_t dt \\ &\geq \int_0^\zeta \int_{\partial B_t^M(p_j)} (n - Cr_j) dA_t dt \\ &\geq \int_0^\zeta (n - Ct) \int_{\partial B_t^M(p_j)} dA_t dt \\ &\geq (n - C\zeta) V_j(\zeta), \end{aligned}$$

where $V_j(\zeta)$ denotes the volume of $B_\zeta^M(p_j)$. Therefore,

$$(2.2) \quad (n - C\zeta) V_j(\zeta) \leq \zeta A_j(\zeta).$$

Since

$$V_j'(\zeta) = \frac{d}{d\sigma} \int_{B_\zeta^M(p_j)} d\sigma = \frac{d}{d\sigma} \int_0^\zeta \int_{\partial B_t^M(p_j)} |\nabla r_j|^{-1} dA_t dt = \int_{\partial B_\zeta^M(p_j)} |\nabla r_j|^{-1} dA \geq A_j(\zeta),$$

then

$$(n - C\zeta) V_j(\zeta) \leq \zeta V_j'(\zeta).$$

Thus,

$$(2.3) \quad \frac{d}{d\sigma} \log V_j(\zeta) = \frac{V_j'(\zeta)}{V_j(\zeta)} \geq \frac{n}{\zeta} - C.$$

By integrating (2.3) from $\varepsilon > 0$ to ζ , we obtain

$$\log V_j(\zeta) - \log V_j(\varepsilon) \geq n \log \zeta - n \log \varepsilon - C(\zeta - \varepsilon),$$

that is

$$\frac{V_j(\zeta)}{V_j(\varepsilon)} \geq \frac{\zeta^n}{\varepsilon^n} e^{-C(\zeta - \varepsilon)}.$$

Now observing that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{V_j(\varepsilon)}{\varepsilon^n} = \omega_n,$$

we obtain

$$V_j(\zeta) \geq \omega_n \zeta^n e^{-C\zeta}$$

for $0 < \zeta \leq a/2$. Thus, we conclude that

$$\begin{aligned} Vol_f(M) &= \int_M e^{-f} d\sigma \geq \sum_{j=1}^{\infty} \int_{B_{a/2}^M(p_j)} e^{-f} d\sigma \\ &\geq \left(\inf_{\overline{B_{2R}^M}(o)} e^{-f} \right) \sum_{j=1}^{\infty} V_j(a/2) = +\infty. \end{aligned}$$

This contradicts the assumption of the finite weighted volume of M . Therefore, M^n is a proper hypersurface of \overline{M}_f^m . \square

Definition 2.1. *The function $f \in C^\infty(\overline{M})$ is said convex if the Hessian of f is non-negative, this is $\overline{\nabla}^2 f \geq 0$.*

Remark 2.1. *Let \overline{M}_f be a complete gradient Ricci soliton satisfying $\overline{Ric}_f = \frac{1}{2}g$. In this case, Cao and Zhou [16] showed that by translating f*

$$(2.4) \quad \overline{R} + |\overline{\nabla} f|^2 - f = 0 \quad \text{e} \quad \overline{R} \geq 0.$$

Thus, it follows from the equations in (2.4) that

$$(2.5) \quad |\overline{\nabla} f|^2 \leq f.$$

In addition, there exists constants $c_1, c_2 \in \mathbb{R}$ such that

$$(2.6) \quad \frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2,$$

where $r(x) = \text{dist}_{\overline{M}}(x, o)$ is the distance of $x \in \overline{M}$ to a fixed point $o \in \overline{M}$. The constant c_2 depends only on the dimension of the manifold and c_1 depends on the geometry of g on unit ball center in o (see [16], Lemma 2.1, Lemma 2.2 and Theorem 1.1). In [9], Munteanu and Wang showed that the inequalities in (2.6) are true only assuming that $\overline{Ric}_f \geq \frac{1}{2}g$ and $|\overline{\nabla} f|^2 \leq f$.

Proof of Proposition 1.2. By hypothesis, $\overline{Ric}_f = \frac{1}{2}g$. Thus, it follows from Remark 2.1 that

$$\overline{R} + |\overline{\nabla} f|^2 - f = 0, \quad \overline{R} + \overline{\Delta} f = \frac{m}{2}, \quad \text{and} \quad \overline{R} \geq 0,$$

and we have that

$$\overline{\Delta} f - |\overline{\nabla} f|^2 + f = \frac{m}{2} \quad \text{and} \quad |\overline{\nabla} f|^2 \leq f.$$

By being f a convex function, we obtain

$$\begin{aligned}
\Delta_f f + f &= \Delta f - |\nabla f|^2 + f = \bar{\nabla}^2 f(e_i, e_i) + \langle \mathbf{H}, \bar{\nabla} f^\perp \rangle - |\nabla f|^2 + f \\
&= \bar{\Delta} f - \sum_{i=n+1}^m \bar{\nabla}^2 f(\eta_i, \eta_i) + \langle \mathbf{H}_f, \bar{\nabla} f^\perp \rangle - |\bar{\nabla} f^\perp|^2 - |\nabla f|^2 + f \\
&= \bar{\Delta} f - \sum_{i=n+1}^m \bar{\nabla}^2 f(\eta_i, \eta_i) + \langle \mathbf{H}_f, \bar{\nabla} f^\perp \rangle - |\bar{\nabla} f|^2 + f \\
&\leq \frac{m}{2} + C,
\end{aligned}$$

where $C = \sup_{x \in M} \langle \mathbf{H}_f, \bar{\nabla} f^\perp \rangle < \infty$.

Observe that

$$(2.7) \quad \frac{1}{4}(r(x) - c)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c)^2,$$

where c is a constant (see Remark 2.1, inequalities in (2.6)). Hence, we can conclude that f is proper on \bar{M} . Since, by hypothesis, $x : M \rightarrow \bar{M}_f$ is a proper immersion, then $f|_M : M \rightarrow \mathbb{R}$ is a proper function.

Therefore, it follows from Theorem 1.1 de [4] that M has finite weighted volume and Euclidean volume growth of the sub-level set of the potential function f . \square

3. THE f -STABILITY INDEX OF THE CONSTANT WEIGHTED MEAN CURVATURE HYPERSURFACES

In this section, we will prove Theorem 1.1, Corollary 1.1, and Theorem 1.2. Which are results about the f -stability index of a constant weighted mean curvature hypersurface with finite weighted volume and isometrically immersed in a gradient Ricci soliton that admits at least one parallel field globally defined. For this, we are going to give some definitions and known results.

Definition 3.1. *We say that a vector field $X \in T\bar{M}$ is parallel if*

$$\bar{\nabla}_Y X = 0$$

for all vector fields $Y \in T\bar{M}$.

Proposition 3.1. *Let \bar{M}_f^{n+1} be a gradient Ricci soliton satisfying $\bar{\text{Ric}}_f = kg$ and X a parallel vector field on \bar{M}_f^{n+1} . If M^n is a constant weighted mean curvature hypersurface immersed isometrically immersed in \bar{M}_f^{n+1} , then*

$$L_f \langle X, \eta \rangle = k \langle X, \eta \rangle$$

and

$$\Delta_f \langle X, \eta \rangle^2 = -2|A|^2 \langle X, \eta \rangle^2 + 2|AX^\top|^2,$$

where η is the unit normal vector field to M .

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be a geodesic orthonormal frame on M . By hypothesis, $H_f = C$, this is $H = \langle \bar{\nabla} f, \eta \rangle + C$, where C is a real constant. Thus,

$$\begin{aligned} \nabla H &= \sum_{i=1}^n e_i \langle H \rangle e_i = \sum_{i=1}^n e_i \langle \langle \bar{\nabla} f, \eta \rangle \rangle e_i = \sum_{i=1}^n \langle \bar{\nabla}_{e_i} \bar{\nabla} f, \eta \rangle e_i + \sum_{i=1}^n \langle \bar{\nabla} f, \bar{\nabla}_{e_i} \eta \rangle e_i \\ (3.1) \quad &= \sum_{i=1}^n \langle \bar{\nabla}_{e_i} \bar{\nabla} f, \eta \rangle e_i - \sum_{i=1}^n \langle A(e_i, e_j), \eta \rangle \langle \bar{\nabla} f, e_j \rangle e_i. \end{aligned}$$

To $u = \langle X, \eta \rangle$ and $a_{ij} = \langle Ae_i, e_j \rangle$, we have

$$(3.2) \quad \nabla u = \sum_{j=1}^n e_j \langle u \rangle e_j = \sum_{j=1}^n \langle \bar{\nabla}_{e_j} \eta, X \rangle e_j = - \sum_{i,j=1}^n a_{ji} \langle e_i, X \rangle e_j.$$

It follows from (3.1) and (3.2),

$$\langle \nabla H, X \rangle = \sum_{i=1}^n \langle \bar{\nabla}_{e_i} \bar{\nabla} f, \eta \rangle \langle e_i, X \rangle - \sum_{i,j=1}^n a_{ij} \langle \bar{\nabla} f, e_j \rangle \langle e_i, X \rangle = \langle \bar{\nabla}_{X^\top} \bar{\nabla} f, \eta \rangle + \langle \bar{\nabla} f, \nabla u \rangle.$$

Moreover,

$$e_i(u) = \langle \bar{\nabla}_{e_i} \eta, X \rangle = - \sum_{j=1}^n a_{ij} \langle e_j, X \rangle$$

and, by deriving the previous expression and observing that $\nabla_{e_k} e_j = 0$, we obtain

$$e_k(e_i(u)) = - \sum_{j=1}^n (a_{ij,k} \langle e_j, X \rangle + a_{ij} \langle X, \bar{\nabla}_{e_k} e_j \rangle) = - \sum_{j=1}^n a_{ij,k} \langle e_j, X \rangle - \sum_{j=1}^n a_{ij} a_{kj} \langle X, \eta \rangle.$$

It follows from Codazzi equation that

$$\bar{R}(e_j, e_k) e_i^\perp = (a_{ki,j} - a_{ji,k}) \eta,$$

that is

$$\langle \bar{R}(e_j, e_k) e_i, \eta \rangle = a_{ki,j} - a_{ji,k}.$$

Hence,

$$e_k(e_i(u)) = - \sum_{j=1}^n a_{ki,j} \langle e_j, X \rangle + \sum_{j=1}^n \langle \bar{R}(e_j, e_k) e_i, \eta \rangle \langle e_j, X \rangle - \sum_{j=1}^n a_{ij} a_{kj} \langle X, \eta \rangle$$

and

$$\begin{aligned}
\Delta u &= \sum_{i=1}^n e_i(e_i(u)) = - \sum_{i,j=1}^n a_{ii,j} \langle e_j, X \rangle + \sum_{i,j=1}^n \langle \bar{R}(e_j, e_i) e_i, \eta \rangle \langle e_j, X \rangle - \sum_{i,j=1}^n a_{ij} a_{ij} \langle X, \eta \rangle \\
&= \langle \nabla H, X \rangle + \sum_{i=1}^n \langle \bar{R}(X^\top, e_i) e_i, \eta \rangle - |A|^2 \langle X, \eta \rangle \\
(3.3) \quad &= \langle \bar{\nabla}_{X^\top} \bar{\nabla} f, \eta \rangle + \langle \bar{\nabla} f, \nabla u \rangle + \overline{Ric}(X^\top, \eta) - |A|^2 u.
\end{aligned}$$

On the other hand,

$$(3.4) \quad 0 = \frac{1}{2} \langle X^\top, \eta \rangle = \overline{Ric}_f(X^\top, \eta) = \overline{Ric}(X^\top, \eta) + \bar{\nabla}^2 f(X^\top, \eta) = \overline{Ric}(X^\top, \eta) + \langle \bar{\nabla}_{X^\top} \bar{\nabla} f, \eta \rangle.$$

Therefore, it follows from (3.3) and (3.4),

$$\begin{aligned}
\Delta_f u &= \Delta u - \langle \nabla f, \nabla u \rangle \\
&= \langle \bar{\nabla}_{X^\top} \bar{\nabla} f, \eta \rangle + \langle \bar{\nabla} f, \nabla u \rangle - \langle \bar{\nabla}_{X^\top} \bar{\nabla} f, \eta \rangle - |A|^2 u - \langle \nabla f, \nabla u \rangle \\
(3.5) \quad &= -|A|^2 u,
\end{aligned}$$

by implying that

$$L_f u = \Delta_f u + |A|^2 u + k u = k u.$$

Moreover, by using the equality (3.5), we have that

$$\Delta_f u^2 = 2u \Delta_f u + 2|\nabla u|^2 = -2|A|^2 u^2 + 2|AX^\top|^2.$$

□

Let $\mathcal{P}_{\bar{M}}$ be the set of all tangent vector fields to \bar{M} which are parallel and globally defined.

Example 3.1. *O shrinking Gaussian soliton $(\mathbb{R}^{n+1}, g_{can}, e^{-|x|^2/2})$ has exactly $n+1$ parallel vector fields linearly independent and globally defined on \mathbb{R}^{n+1} . Other example is the shrinking cylinder soliton $(\mathbb{S}^{n+1-k} \times \mathbb{R}^k, g, e^{-f})$, $k \geq 1$, with metric*

$$g = 2(n-k-1)g_{\mathbb{S}^{n-k}} + g_{\mathbb{R}^k}$$

and potential function

$$f(\theta, x) = \frac{|x|^2}{2}, \quad \theta \in \mathbb{S}^{n-k}, \quad x \in \mathbb{R}^k.$$

In this example, we have that

$$\dim \mathcal{P}_{\mathbb{S}^{n+1-k} \times \mathbb{R}^k} = k.$$

Definition 3.2. *The vector subspace of $C^\infty(M)$ generated by $E \subset C^\infty(M)$, denoted by $\text{Span } E$, is the set of all the linear combinations of the elements of E .*

Lemma 3.1. *Let \overline{M}_f^{n+1} be a shrinking gradient Ricci soliton satisfying $\overline{\text{Ric}}_f = kg$ and M^n be a constant weighted mean curvature hypersurface isometrically immersed in \overline{M}_f^{n+1} . If M is compact, then I_f is negative defined in the*

$$\text{Span} \{1, \langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}.$$

Moreover,

$$\int_M |A|^2 \langle X, \eta \rangle e^{-f} d\sigma = 0.$$

Proof. It follows from Proposition 3.1 that the function $u = \langle X, \eta \rangle$, with $X \in \mathcal{P}_{\overline{M}_f}$, satisfies $L_f u = ku$. Thus, since M is compact, we have

$$\begin{aligned} \int_M k u e^{-f} d\sigma &= \int_M L_f u e^{-f} d\sigma = \int_M (\Delta_f u + |A|^2 u + k u) e^{-f} d\sigma \\ &= \int_M |A|^2 u e^{-f} d\sigma + \int_M k u e^{-f} d\sigma. \end{aligned}$$

Therefore,

$$(3.6) \quad \int_M |A|^2 u e^{-f} d\sigma = 0.$$

Observe that

$$I_f(1, 1) = - \int_M 1 L_f 1 e^{-f} d\sigma = - \int_M (|A|^2 + k) e^{-f} d\sigma$$

and

$$I_f(u, u) = - \int_M u L_f u e^{-f} d\sigma = -k \int_M u^2 e^{-f} d\sigma.$$

Therefore,

$$\begin{aligned} I_f(c_0 + u, c_0 + u) &= I_f(c_0, c_0) + I_f(u, u) + 2I_f(c_0, u) \\ &= - \int_M [c_0^2 |A|^2 + k c_0^2 + k u^2 + 2c_0 (\Delta_f u + |A|^2 u + k u)] e^{-f} d\sigma \\ &= - \int_M [c_0^2 |A|^2 + k c_0^2 + k u^2 + 2c_0 k u] e^{-f} d\sigma \\ &= -c_0^2 \int_M |A|^2 e^{-f} d\sigma - k \int_M (c_0 + u)^2 e^{-f} d\sigma < 0, \end{aligned}$$

where $u = \langle X, \eta \rangle$. This shows that I_f is negative defined in $\text{Span} \{1, \langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}$. \square

Remark 3.1. *Supposing that M^n has finite weighted volume and putting $\alpha = - \int_M \langle X, \eta \rangle e^{-f} d\sigma$, we can conclude that*

$$\int_M (\alpha + \langle X, \eta \rangle) e^{-f} d\sigma = 0.$$

Therefore,

$$\mathcal{F} \cap \text{Span} \{1, \langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\} \neq \emptyset.$$

Now, let's look at the noncompact manifolds. For this, we will consider the functions that have compact support in M .

Proposition 3.2. *Let \overline{M}_f^{n+1} be a weighted manifold and M^n be a noncompact hypersurface isometrically immersed in \overline{M}_f^{n+1} . Then*

$$I_f(\phi u, \phi u) = - \int_M \phi^2 u L_f u e^{-f} d\sigma + \int_M |\nabla \phi|^2 u^2 e^{-f} d\sigma,$$

where $\phi \in C_c^\infty(M)$, $u \in C^\infty(M)$, and η denotes the unit normal field on M . Moreover, if $\overline{Ric}_f = kg$ and M has constant weighted mean curvature, then

$$(3.7) \quad \int_M \phi^2 |A|^2 \langle X, \eta \rangle e^{-f} d\sigma = -2 \int_M \phi \langle \nabla \phi, AX^\top \rangle e^{-f} d\sigma,$$

where X is a parallel vector field on \overline{M}_f^{n+1} .

Proof. Note that

$$\begin{aligned} I_f(\phi u, \phi u) &= - \int_M (\phi u) L_f (\phi u) e^{-f} d\sigma \\ &= - \int_M [(\phi u) \Delta_f (\phi u) + (|A|^2 + \overline{Ric}_f(\eta, \eta)) \phi^2 u^2] e^{-f} d\sigma \\ &= - \int_M [\phi^2 u \Delta_f u + \phi u^2 \Delta_f \phi + 2\phi u \langle \nabla \phi, \nabla u \rangle + (|A|^2 + \overline{Ric}_f(\eta, \eta)) \phi^2 u^2] e^{-f} d\sigma, \end{aligned}$$

As ϕ has compact support, then

$$0 = \int_M \operatorname{div}(\phi u^2 e^{-f} \nabla \phi) d\sigma = \int_M (\phi u^2 \Delta_f \phi + 2u \phi \langle \nabla u, \nabla \phi \rangle + u^2 |\nabla \phi|^2) e^{-f} d\sigma.$$

Therefore, by using the last two expressions, we obtain

$$\begin{aligned} I_f(\phi u, \phi u) &= - \int_M [\phi^2 u \Delta_f u - |\nabla \phi|^2 u^2 + (|A|^2 + \overline{Ric}_f(\eta, \eta)) \phi^2 u^2] e^{-f} d\sigma \\ &= - \int_M \phi^2 u L_f u e^{-f} d\sigma + \int_M |\nabla \phi|^2 u^2 e^{-f} d\sigma. \end{aligned}$$

Moreover, supposing $\overline{Ric}_f = kg$, H_f is constant, and X is a parallel field on \overline{M}_f , we obtain

$$\begin{aligned} \int_M \phi^2 k \langle X, \eta \rangle e^{-f} d\sigma &= \int_M \phi^2 L_f \langle X, \eta \rangle e^{-f} d\sigma \\ &= - \int_M \langle \nabla \phi^2, \nabla \langle X, \eta \rangle \rangle e^{-f} d\sigma + \int_M (|A|^2 + k) \phi^2 \langle X, \eta \rangle e^{-f} d\sigma. \end{aligned}$$

Hence,

$$\int_M |A|^2 \phi^2 \langle X, \eta \rangle e^{-f} d\sigma = \int_M \langle \nabla \phi^2, \nabla \langle X, \eta \rangle \rangle e^{-f} d\sigma = -2 \int_M \phi \langle \nabla \phi, AX^\top \rangle e^{-f} d\sigma.$$

□

Lemma 3.2. *Let \overline{M}_f^{n+1} be a shrinking gradient Ricci soliton satisfying $\overline{Ric}_f = kg$ and M^n be a constant weighted mean curvature hypersurface isometrically immersed in \overline{M}_f^{n+1} . If M is noncompact and finite weighted volume, there exists $\phi \in C_c^\infty(M)$ such that I_f is negative defined in ϕV and $\dim(\phi V) = \dim V$, where*

$$V = \text{Span} \left\{ 1, \langle X, \eta \rangle : X \text{ is a parallel field on } \overline{M}_f^{n+1} \right\}.$$

Moreover, by assuming that $\int_M |A|^2 e^{-f} d\sigma < \infty$, we have

$$\int_M |A|^2 \langle X, \eta \rangle e^{-f} d\sigma = 0.$$

Proof. Let $u = c_0 + \langle X, \eta \rangle$, where c_0 is a real constant and X is a parallel field on \overline{M}_f^{n+1} . As H_f is constant, then by Proposition 3.1, we have

$$\begin{aligned} L_f u &= \Delta_f u + (|A|^2 + k) u = L_f \langle X, \eta \rangle + (|A|^2 + k) c_0 \\ (3.8) \quad &= k \langle X, \eta \rangle + (|A|^2 + k) c_0. \end{aligned}$$

It follows from Proposition 3.2 and equality (3.8) that

$$\begin{aligned} I_f(\phi u, \phi u) &= - \int_M \phi^2 u L_f u e^{-f} d\sigma + \int_M |\nabla \phi|^2 u^2 e^{-f} d\sigma \\ &= - \int_M \phi^2 u (k \langle X, \eta \rangle + (|A|^2 + k) c_0) e^{-f} d\sigma + \int_M |\nabla \phi|^2 u^2 e^{-f} d\sigma \\ &= -k \int_M \phi^2 u^2 e^{-f} d\sigma - \int_M \phi^2 |A|^2 c_0^2 e^{-f} d\sigma - \int_M \phi^2 |A|^2 c_0 \langle X, \eta \rangle e^{-f} d\sigma \\ &\quad + \int_M |\nabla \phi|^2 u^2 e^{-f} d\sigma. \end{aligned}$$

Now, by using once more the Proposition 3.2 and Cauchy-Schwarz's inequality,

$$\begin{aligned} \left| \int_M \phi^2 |A|^2 c_0 \langle X, \eta \rangle e^{-f} d\sigma \right| &= 2 \left| \int_M \phi \langle \nabla \phi, AX^\top \rangle c_0 e^{-f} d\sigma \right| \\ &\leq 2 \int_M |\phi| |\nabla \phi| |A| |X^\top| |c_0| e^{-f} d\sigma \\ &\leq \int_M \phi^2 |A|^2 c_0^2 e^{-f} d\sigma + \int_M |\nabla \phi|^2 |X^\top|^2 e^{-f} d\sigma. \end{aligned}$$

Therefore,

$$(3.9) \quad I_f(\phi u, \phi u) \leq -k \int_M \phi^2 u^2 e^{-f} d\sigma + \int_M |\nabla \phi|^2 (u^2 + |X^\top|^2) e^{-f} d\sigma.$$

Let $r(x)$ be the extrinsic distance from $x \in M$ to a fixed point $o \in \overline{M}_f$. For $R > 0$ sufficiently large, define the function $\phi_R : M \rightarrow \mathbb{R}$ such that

$$(3.10) \quad \phi_R(x) = \begin{cases} 1, & r(x) \leq R; \\ \frac{2R - r(x)}{R}, & R \leq r(x) \leq 2R; \\ 0, & r(x) \geq 2R. \end{cases}$$

Observe that $|\nabla \phi_R| \leq 1/R$ and $\phi_R \in C_c^\infty(M)$ because M is proper (see Proposition 1.1). Now, by substituting $\phi = \phi_R$ in the inequality (3.9), we obtain

$$I_f(\phi_R u, \phi_R u) \leq -k \int_M \phi_R^2 u^2 e^{-f} d\sigma + \frac{|X|^2 + c_0^2 + 2|c_0||X|}{R^2} \int_{M \cap (B_{2R} \setminus B_R)} e^{-f} d\sigma,$$

(recall that $|X|$ is constant because X is parallel). Since $\text{Vol}_f(M) < \infty$ and $|X|$ is constant, then for each $u \in V$ there exists R_u sufficiently large such that

$$I_f(\phi_{R_u} u, \phi_{R_u} u) < 0.$$

Let's find a function $\phi \in C_c^\infty(M)$ that is not dependent of the function u . In fact, we consider the subset

$$S = \left\{ u \in V : \int_M u^2 e^{-f} d\sigma = 1 \right\}.$$

Note that $V \subset L_f^2(M)$ is a subspace of finite dimension smaller than or equal to $\dim \mathcal{P}_{\overline{M}_f} + 1$. Hence, S is a compact set in $L_f^2(M)$. Thus, there exists a positive real number R_0 such that any function $u \in S$ vanishes identically on $M \cap B_{R_0}^{\overline{M}_f}(o)$. Otherwise, we could get a sequence $R_j \rightarrow \infty$ of positive numbers so that for each j exists $u_j \in S$ with $u_j \equiv 0$ on $M \cap B_{R_0}^{\overline{M}_f}(o)$. Hence, we would have

$$u = \lim_{j \rightarrow \infty} u_j \in S$$

and $u \equiv 0$ on M . However, this is not possible because if $u \in S$, then

$$\int_M u^2 e^{-f} d\sigma = 1.$$

Therefore, to R sufficiently large and $R \geq R_0$, and for any function $u \in S$, we have

$$I_f(\phi_R u, \phi_R u) \leq -k \int_M \phi_R^2 u^2 e^{-f} d\sigma + \frac{|X|^2 + c_0^2 + 2|c_0||X|}{R^2} \int_{M \cap (B_{2R} \setminus B_R)} e^{-f} d\sigma < 0$$

because

$$M(R) = \int_M \phi_R^2 u^2 e^{-f} d\sigma > 0$$

is an increasing function on R , and as $Vol_f(M) < \infty$,

$$\lim_{R \rightarrow \infty} \frac{|X|^2 + c_0^2 + 2|c_0||X|}{R^2} \int_{M \cap (B_{2R} \setminus B_R)} e^{-f} d\sigma = 0.$$

Now, we can find $\phi = \phi_R$ independent of u such that $I_f(\phi u, \phi u) < 0$ for all $u \in S$. If $u \in V$, then $\frac{1}{|u|_{L_f^2}} u \in S$, and thus, for $u \neq 0$,

$$I_f(\phi u, \phi u) = |u|_{L_f^2}^2 I_f \left(\phi \frac{u}{|u|_{L_f^2}}, \phi \frac{u}{|u|_{L_f^2}} \right) < 0.$$

Now, we will show that $\dim V = \dim(\phi V)$. In fact, let $\{u_1, u_2, \dots, u_s\}$ be a orthonormal basis to the vector subspace $V \subset L_f^2(M)$. For the function ϕ built here, we have that $u_i \neq 0$ on $M \cap \bar{B}_R^{\bar{M}^f}(o)$. Therefore, $\{\phi u_1, \phi u_2, \dots, \phi u_s\}$ is linearly independent, so we can conclude that $\dim(\phi V) = \dim V$.

Finally, it follows from Proposition 3.2, equation (3.7) with $\phi = \sqrt{\phi_j}$, and Cauchy-Schwarz inequality that

$$\begin{aligned} \left| 2 \int_M \phi_j |A|^2 \langle X, \eta \rangle e^{-f} d\sigma \right| &= \left| -2 \int_M \langle \nabla \phi_j, A X^\top \rangle e^{-f} d\sigma \right| \\ &\leq \left(\int_M |\nabla \phi_j|^2 e^{-f} d\sigma \right)^{\frac{1}{2}} \left(\int_M |A X^\top|^2 e^{-f} d\sigma \right)^{\frac{1}{2}}. \end{aligned}$$

Now, putting $R = j$ in the function defined in (3.10) and reviewing that $|X|$ is constant, we get

$$(3.11) \quad \left| 2 \int_M \phi_j |A|^2 \langle X, \eta \rangle e^{-f} d\sigma \right| \leq \frac{|X|}{j} \left(\int_{M \cap (B_{2j} \setminus B_j)} e^{-f} d\sigma \right)^{\frac{1}{2}} \left(\int_M |A|^2 e^{-f} d\sigma \right)^{\frac{1}{2}}.$$

By hypothesis M has finite weighted volume and $\int_M |A|^2 e^{-f} d\sigma < \infty$, hence the right-hand side of (3.11) tends to zero as $j \rightarrow \infty$, and we can conclude that

$$(3.12) \quad \lim_{j \rightarrow \infty} \int_M \phi_j |A|^2 \langle X, \eta \rangle e^{-f} d\sigma = 0.$$

Since

$$\lim_{j \rightarrow \infty} (\phi_j |A|^2 \langle X, \eta \rangle)(x) = (|A|^2 \langle X, \eta \rangle)(x)$$

for each $x \in M$,

$$|\phi_j |A|^2 \langle X, \eta \rangle| \leq |A|^2 |\langle X, \eta \rangle|$$

and

$$0 \leq \int_M |A|^2 |\langle X, \eta \rangle| e^{-f} d\sigma \leq |X|^2 \int_M |A|^2 e^{-f} d\sigma < \infty,$$

then using the dominated convergence theorem and expression (3.12), we get

$$\int_M |A|^2 \langle X, \eta \rangle e^{-f} d\sigma = 0.$$

□

Proof of Theorem 1.1. Let $V \equiv \text{Span} \{1, \langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}$. Since $1 \notin \{\langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}$ by hypothesis,

$$\dim V = 1 + \dim\{\langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}.$$

By Lemmas 3.1 and 3.2, there exists a function $\phi \in C_c^\infty(M)$ such that $\dim \phi V = \dim V$ and I_f is negative defined in ϕV . In compact hypersurfaces, choose $\phi \equiv 1$. Recall that the $\text{Ind}_f(M)$ is the maximal dimension of a subspace of $\mathcal{F} \cap C_c^\infty(M)$ which I_f is negative defined, with $\mathcal{F} = C_c^\infty(M)$ if M is a f -minimal and

$$\mathcal{F} = \left\{ u \in C_c^\infty(M); \int_M u e^{-f} d\sigma = 0 \right\}$$

if M has constant weighted mean curvature.

Now, observe that

$$\dim\{\langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\} \leq \dim(\mathcal{F} \cap \phi V).$$

In fact, $|X|$ is constant because X is a parallel field, and since the weighted volume of M is finite, we obtain that $\int_M \phi \langle X, \eta \rangle e^{-f} d\sigma \leq |X| \int_M \phi e^{-f} d\sigma < \infty$. Hence, there exists a real number c_0 satisfying

$$\int_M \phi (c_0 + \langle X, \eta \rangle) e^{-f} d\sigma = 0.$$

Therefore, $\phi(c_0 + \langle X, \eta \rangle) \in \mathcal{F} \cap \phi V$ whenever $\langle X, \eta \rangle \in \{\langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}$. Since I_f is negative defined in $\mathcal{F} \cap \phi V$, we conclude that

$$\dim\{\langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\} \leq \dim(\mathcal{F} \cap \phi V) \leq \text{Ind}_f(M).$$

Consider the linear transformation $T : \mathcal{P}_{\overline{M}_f} \rightarrow C^\infty(M)$ defined by $T(X) = \langle X, \eta \rangle$. Now applying the kernel and image theorem, that turns

$$\begin{aligned} \dim \mathcal{P}_{\overline{M}_f} &= \dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\} + \dim\{\langle X, \eta \rangle; X \in \mathcal{P}_{\overline{M}_f}\} \\ (3.13) \quad &\leq \dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\} + \text{Ind}_f(M). \end{aligned}$$

Otherwise, if $1 \in \{\langle X, \eta \rangle : X \in \mathcal{P}_{\overline{M}_f}\}$, $L_f 1 = k$ by Proposition 3.1, and by definition $L_f 1 = |A|^2 + k$. Therefore, $|A|^2 \equiv 0$. □

Proof of Corollary 1.1. Note that

$$(3.14) \quad \dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\} \leq \dim \mathcal{P}_{\overline{M}_f} - 1$$

as long as we assume that exists a field $X_0 \in \mathcal{P}_{\overline{M}_f}$ such that $\langle X_0, \eta \rangle \not\equiv 0$. It follows from Theorem 1.1 and from the inequality (3.14) that

$$\dim \mathcal{P}_{\overline{M}_f} - \text{Ind}_f M \leq \dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\} \leq \dim \mathcal{P}_{\overline{M}_f} - 1.$$

Therefore,

$$\text{Ind}_f M \geq 1.$$

Now supposing $\text{Ind}_f M = 1$, we have

$$\dim \mathcal{P}_{\overline{M}_f} - 1 \leq \dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\} \leq \dim \mathcal{P}_{\overline{M}_f} - 1.$$

Therefore,

$$\dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\} = \dim \mathcal{P}_{\overline{M}_f} - 1.$$

□

Now, let's obtain a necessary condition for a constant weighted mean curvature hypersurface M^n with finite weighted volume satisfies the following equality:

$$\text{Ind}_f(M) = \dim \mathcal{P}_{\overline{M}_f} - \dim\{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\}.$$

For this, we will prove some lemmas, they are adaption of known results.

The weighted Laplacian operator Δ_f is associated to $e^{-f}d\sigma$ as well as Δ is associated to $d\sigma$. This is viewed in the following Green's theorem version for wighted Laplacian:

Lemma 3.3. *Let $\Omega \subset M$ be a compact set and let $u, v \in C^\infty(\Omega)$, then*

$$(3.15) \quad \int_{\Omega} (u \Delta_f v) e^{-f} d\sigma + \int_{\Omega} \langle \nabla u, \nabla v \rangle e^{-f} d\sigma = \int_{\partial\Omega} u \nu(v) e^{-f} d\partial\Omega, \quad \text{and}$$

$$(3.16) \quad \int_{\Omega} (u \Delta_f v - v \Delta_f u) e^{-f} d\sigma = \int_{\partial\Omega} (u \nu(v) - v \nu(u)) e^{-f} d\partial\Omega,$$

where ν denotes the exterior unit normal to Ω along $\partial\Omega$.

Proof. In fact,

$$\begin{aligned} \text{div}(e^{-f} u \nabla v) &= e^{-f} u \text{div}(\nabla v) + \langle \nabla(e^{-f} u), \nabla v \rangle \\ &= e^{-f} u \Delta v - e^{-f} u \langle \nabla f, \nabla v \rangle + e^{-f} \langle \nabla u, \nabla v \rangle \\ &= e^{-f} u \Delta_f v + e^{-f} \langle \nabla u, \nabla v \rangle. \end{aligned}$$

Integrating both sides from above inequality and applying the divergent theorem to the field $X = e^{-f}u\nabla v$, it follows the equality (3.15). The equality (3.16) is obtained by integrating the difference $u\Delta_f v - v\Delta_f u$ and by applying the equality (3.15). \square

Lemma 3.4 ([15], Corollary 3.10). *Suppose that M is a complete hypersurface without boundary. If u, v are C^2 functions with*

$$(3.17) \quad \int_M (|u\nabla v| + |\nabla u| |\nabla v| + |u\Delta_f v|) e^{-f} d\sigma < \infty,$$

then

$$(3.18) \quad \int_M u(\Delta_f v) e^{-f} d\sigma = - \int_M \langle \nabla v, \nabla u \rangle e^{-f} d\sigma.$$

Let $W^{1,2}(e^{-f}d\sigma)$ be the weighted Sobolev space, which is the set of the functions u on M satisfying

$$\int_M (u^2 + |\nabla u|^2) e^{-f} d\sigma < \infty$$

with the norm

$$\|u\|_{W_f^{1,2}} := \left(\int_M (u^2 + |\nabla u|^2) e^{-f} d\sigma \right)^{\frac{1}{2}}.$$

Fixed the notation $W_f^{1,2} = W_f^{1,2}(e^{-f}d\sigma)$ and $L_f^2 = L_f^2(e^{-f}d\sigma)$, we get

Lemma 3.5. *Let M^n be a constant weighted mean curvature hypersurface isometrically immersed in a gradient Ricci soliton \overline{M}_f that satisfies $\overline{Ric}_f = kg$. Suppose that h is a C^2 function with $L_f h = -\mu h$ for $\mu \in \mathbb{R}$.*

(i) *If $h \in W_f^{1,2}$, then $|A|h \in L_f^2$ and*

$$(3.19) \quad \int_M |A|^2 h^2 e^{-f} d\sigma \leq \int_M ((1 - k - \mu)h^2 + 2|\nabla h|^2) e^{-f} d\sigma.$$

(ii) *If $h > 0$ and $\phi \in W_f^{1,2}$, then*

$$(3.20) \quad \int_M \phi^2 (2|A|^2 + |\nabla \log h|^2) e^{-f} d\sigma \leq \int_M [4|\nabla \phi|^2 - 2(\mu + k)\phi^2] e^{-f} d\sigma.$$

Proof. Let ϕ be a smooth function with compact support. Note that

$$\Delta_f h^2 = 2|\nabla h|^2 + 2h\Delta_f h$$

and

$$(3.21) \quad \Delta_f h = (L_f - |A|^2 - k)h = -(\mu + |A|^2 + k)h.$$

By Lemma 3.3 and equality (3.21),

$$\begin{aligned}
 \int_M \langle \nabla \phi^2, \nabla h^2 \rangle e^{-f} d\sigma &= - \int_M \phi^2 \Delta_f h^2 e^{-f} d\sigma \\
 (3.22) \qquad \qquad \qquad &= -2 \int_M \phi^2 [|\nabla h|^2 - (\mu + |A|^2 + k) h^2] e^{-f} d\sigma.
 \end{aligned}$$

Assume now that $\phi \leq 1$ and $|\nabla \phi| \leq 1$. Rearranging the terms in (3.22) and using the inequality

$$0 \leq |\phi \nabla h - h \nabla \phi|^2 = \phi^2 |\nabla h|^2 + h^2 |\nabla \phi|^2 - 2\phi h \langle \nabla \phi, \nabla h \rangle \leq |\nabla h|^2 + h^2 - 2\phi h \langle \nabla \phi, \nabla h \rangle,$$

we have

$$\begin{aligned}
 \int_M \phi^2 (2k + 2\mu + 2|A|^2) h^2 e^{-f} d\sigma &= 4 \int_M \phi h \langle \nabla \phi, \nabla h \rangle e^{-f} d\sigma + 2 \int_M \phi^2 |\nabla h|^2 e^{-f} d\sigma \\
 (3.23) \qquad \qquad \qquad &\leq 2 \int_M h^2 e^{-f} d\sigma + 4 \int_M |\nabla h|^2 e^{-f} d\sigma.
 \end{aligned}$$

Finally, consider the intrinsic ball $B_j = B_j(p)$ in M of radius j and center at a fixed point $p \in M$. By applying (3.23) with $\phi = \phi_j$, where ϕ_j is one on B_j and cuts off linearly to zero from ∂B_j to ∂B_{j+1} , letting $j \rightarrow \infty$, and using the monotone convergence theorem we obtain

$$\int_M |A|^2 h^2 e^{-f} d\sigma \leq \int_M ((1 - k - \mu)h^2 + 2|\nabla h|^2) e^{-f} d\sigma.$$

Now, we will prove the inequality (3.20). In fact, $\log h$ is well-defined and

$$\begin{aligned}
 \Delta_f \log h &= \frac{1}{h} \Delta_f h - |\nabla \log h|^2 = \frac{1}{h} L_f h - |A|^2 - k - |\nabla \log h|^2 \\
 (3.24) \qquad \qquad &= -\mu - |A|^2 - k - |\nabla \log h|^2.
 \end{aligned}$$

Let ψ be a function with compact support. Thus, it follows from Lemma 3.3 and expression (3.24) that

$$\begin{aligned}
 \int_M \langle \nabla \psi^2, \nabla \log h \rangle e^{-f} d\sigma &= - \int_M \psi^2 (\Delta_f \log h) e^{-f} d\sigma \\
 (3.25) \qquad \qquad \qquad &= \int_M \psi^2 (\mu + |A|^2 + k + |\nabla \log h|^2) e^{-f} d\sigma.
 \end{aligned}$$

Combining (3.25) with the following inequality

$$\langle \nabla \psi^2, \nabla \log h \rangle \leq 2|\nabla \psi|^2 + \frac{1}{2}\psi^2 |\nabla \log h|^2,$$

gives us this

$$(3.26) \qquad \int_M \psi^2 (2|A|^2 + |\nabla \log h|^2) e^{-f} d\sigma \leq \int_M (4|\nabla \psi|^2 - 2\mu\psi^2 - 2k\psi^2) e^{-f} d\sigma.$$

Let ψ_j be one on B_j and cut off linearly to zero from ∂B_j to ∂B_{j+1} . Since $\phi \in W^{1,2}$, applying (3.26) with $\psi = \psi_j \phi$, and letting $j \rightarrow \infty$, using the monotone convergence theorem, we get

$$\int_M \phi^2 (2|A|^2 + |\nabla \log h|^2) e^{-f} d\sigma \leq \int_M [4|\nabla \phi|^2 - 2(\mu + k)\phi^2] e^{-f} d\sigma.$$

□

Remark 3.2. Lemma 3.5 was obtain by Colding and Minicozzi II (see [15], Lemma 9.15) to a complete noncompact hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ without boundary that satisfies

$$H = \frac{\langle x, \eta \rangle}{2},$$

where x is position vector.

Through use the Lemmas 3.4 and 3.5, and ideas as the proof of Lemma 9.25 from [15] we have

Lemma 3.6. Let $\mu_1(M)$ be the bottom L_f^2 spectrum of L_f . If $\mu_1(M) \neq -\infty$, then there exists a positive C^2 function u on M with $L_f u = -\mu_1(M)u$. Moreover, if $w \in W_f^{1,2}$ and $L_f w = -\mu_1(M)w$, then $w = Cu$ for some $C \in \mathbb{R}$.

Lemma 3.7. Let M be a complete oriented constant weighed mean curvature hypersurface in gradient Ricci soliton \overline{M}_f . If $\mu_1(M) \neq -\infty$ and $\text{Vol}_f(M) < \infty$, then

$$(3.27) \quad \int_M |A|^2 e^{-f} d\sigma < \infty.$$

Proof. Since $\mu_1(M) \neq -\infty$, there is a C^2 positive function h on M satisfying $L_f h = -\mu_1(M)h$ by Lemma 3.6. Let ϕ_j be the cut off functions such that $|\nabla \phi_j| \leq 1$ and $\phi_j \leq 1$. So $\phi_j \in W^{1,2}(e^{-f} d\sigma)$ because $\text{Vol}_f(M) < \infty$. By Lemma 3.5, equality (3.20), we have

$$\begin{aligned} 2 \int_M \phi_j^2 |A|^2 e^{-f} d\sigma &\leq \int_M \phi_j^2 (2|A|^2 + |\nabla \log h|^2) e^{-f} d\sigma \\ &\leq \int_M (4|\nabla \phi_j|^2 - 2(\mu_1(M) + k)\phi_j^2) e^{-f} d\sigma \\ &\leq (4 + 2|\mu_1(M) + k|) \int_M e^{-f} d\sigma < \infty. \end{aligned}$$

By letting $j \rightarrow \infty$, we obtain the conclusion this lemma by monotone convergence theorem. □

The compact manifolds which admit a parallel vector field with respect to some metric was characterized by Welsh in [13]. Namely, they are the compact fibre bundles over tori with finite structural group. The dimension of the torus can be assumed to be the number of linearly independent parallel vector fields.

The noncompact case has also been solved by Welsh in [12]. If fact,

Proposition 3.3 ([12], Proposition 2.1). *If a Riemannian manifold M admits a complete parallel vector field, then either M is diffeomorphic to the product of a Euclidian space with some other manifold or there is a circle action on M whose orbits are not real homologous to zero.*

Proof of Theorem 1.2. Now suppose that $\dim \mathcal{P}_{\overline{M}_f} = l > 0$ and by hypothesis

$$\text{Ind}_f(M) = \dim \mathcal{P}_{\overline{M}_f} - \dim \{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\}.$$

We can have two cases: either $\text{Ind}_f(M) = \dim \mathcal{P}_{\overline{M}_f}$ or $\text{Ind}_f(M) \neq \dim \mathcal{P}_{\overline{M}_f}$.

Initially, suppose that $\text{Ind}_f(M) = \dim \mathcal{P}_{\overline{M}_f}$, that is $\dim \{X \in \mathcal{P}_{\overline{M}_f} : \langle X, \eta \rangle \equiv 0\} = 0$ and there exists l independent linearly parallel unit vector fields X_1, X_2, \dots, X_l such that $\langle X_j, \eta \rangle \not\equiv 0$ for all $j = 1, 2, \dots, l$. Put

$$u_j = \langle X_j, \eta \rangle \quad \text{for all } j = 1, 2, \dots, l.$$

By Proposition 3.1,

$$L_f u_j = k u_j,$$

where $\overline{\text{Ric}}_f = kg$. Note that

$$(3.28) \quad \int_M u_j^2 e^{-f} d\sigma \leq \int_M e^{-f} d\sigma < \infty$$

because

$$u_j = \langle X_j, \eta \rangle \leq |X_j| |\eta| = 1.$$

Hence, u_1, u_2, \dots, u_l are $L^2(e^{-f} d\sigma)$ eigenfunctions with negative eigenvalue $-k$. Since, by hypothesis,

$$\text{Ind}_f(M) = l > 0,$$

then the bottom $\mu_1(M)$ of the spectrum of L_f satisfies $\mu_1(M) = -k$.

Now, consider a local orthonormal frame $\{e_1, e_2, \dots, e_n\}$ on M . Observe that

$$(3.29) \quad |\nabla u_j|^2 = \sum_{i=1}^n |\nabla_{e_i} u_j|^2 \geq \sum_{i=1}^n \left(\sum_{k=1}^n a_{ik}^2 \right) \left(\sum_{k=1}^n \langle e_k, X_j \rangle^2 \right) \leq |A|^2.$$

Hence, it follows from Lemma 3.7 and inequality (3.29) that

$$\int_M |\nabla u_j|^2 e^{-f} d\sigma \leq \int_M |A|^2 e^{-f} d\sigma < \infty$$

and by using (3.28), we get $u_j \in W^{1,2}(e^{-f} d\sigma)$. By Lemma 3.6, $u_j > 0$ on M without lost of generality. Therefore, it follows from Lemma 3.2 and the integrability of $|A|^2$, that

$$\int_M |A|^2 u_j e^{-f} d\sigma = 0.$$

Thus, $|A| \equiv 0$ on M .

In the case which $\text{Ind}_f(M) \neq \dim \mathcal{P}_{\overline{M}_f}$, we have there exists a $u_0 \equiv 0$, where $u_0 = \langle X_0, \eta \rangle$ and $X_0 \in T\overline{M}_f$ is parallel field with respect to Riemannian connection of $(\overline{M}^{n+1}, \overline{g})$, in particular, $X_0 \in TM$ is parallel with respect to Riemannian connection of (M, g) . By Proposition 3.3, either M is diffeomorphic to the product of a Euclidian space with some other manifold, or there is a circle action on M whose orbits are not real homologous to zero.

□

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